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LETTER TO THE EDITOR

An integrable (2+1)-dimensional generalisation of the Volterra model

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Abstract. An integrable system in two discrete spatial variables and a continuous time is presented. It contains the Volterra model as a particular case. Both the system and their solutions are characterised into the framework of asymptotic modules. Rational and soliton-like solutions are exhibited.

This letter is part of a series devoted to a new method for introducing integrable non-linear equations (NE) and characterising their solutions. Here we are concerned with the following (2+1)-dimensional integrable model in two discrete 'spatial' variables $(n, m) \in \mathbb{Z}^2$ and a continuous 'time' $t \in \mathbb{R}$:

$$(\eta-\xi)(\eta-\xi^{-1})\frac{\partial_t b}{b}=(\eta-\xi^{-1})(1-\xi^2)\frac{\eta b}{\xi^{-1}b}+(\eta-\xi)(1-\xi^{-2})\frac{\xi b}{\eta b} \tag{1}$$

where $b=b(n, m, t)$ is a complex-valued function and ξ, η are the translation operators for the discrete variables

$$\xi g(n, m)=g(n-1, m) \qquad \eta g(n, m)=g(n, m-1).$$

We note that by dropping the m dependence our model reduces to the Volterra equation [1, 2]

$$\partial_t c/c=2(\xi+\xi^{-1})c \qquad c=\xi b_0/b_0. \tag{2}$$

Since (2) is a discrete version of the Korteweg–de Vries equation, (1) may be considered as a discrete analogue of the Kadomtsev–Petviashvili equation.

Broad classes of solutions to (1) can be obtained by means of certain objects called asymptotic modules (AM). As an illustration of this procedure we deduce the one-soliton solution

$$\frac{b(n, m-1, t)}{b(n+1, m, t)}=1+\frac{2\sinh\alpha\,\sinh[\frac{1}{2}(\alpha+\beta)]}{\cosh(\alpha n+\beta m+\omega t)+\cosh[\frac{1}{2}(\alpha-\beta)]} \tag{3}$$

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as well as the rational-like solution

$$b(n, m, t) = 1 - \frac{2[1+(-1)^{n+m}]}{1+2[n+(\gamma^2-1)(\gamma^2+1)^{-1}m-2(\gamma^2+\gamma^{-2})t]}. \tag{4}$$

In this letter proofs are only outlined. A full account of our work will be published elsewhere with the investigation of a more general discrete (2+1)-dimensional model and the analysis of the continuous limit.

Generally speaking an AM is a set of matrix-valued functions $\varphi(k, x, y, \ldots, t)$ with a difference-differential structure in the discrete or continuous parameters $(x, y, \ldots, t)$ and an 'asymptotic' analytic structure in k around some poles $K_1, K_2, \ldots$, on the Riemann sphere. These two structures are coupled thanks to a $\mathscr{D}$-module structure, where $\mathscr{D}$ is some ring of linear difference-differential operators. Then some NE connecting some 'asymptotic' coefficients depending on $x, y, \ldots, t$ may arise as a compatibility condition between the difference-differential and the analytic structures. AM of the 'same type' lead to the introduction of the same NE for which each of them affords one solution. In 'good' cases these NE are purely difference-differential and involve a small number of unknown functions. More information on the AM scheme is given in [3].

In the present work, by an AM we mean a set $\mathscr{F}$ of complex-valued functions $\varphi(k, n, m, t) = \hat{\varphi}(k, n, m, t) f_0(k, n, m, t)$ where

$$f_0(k, n, m, t) = k^{-n}(k+k^{-1})^{-m} \exp[(k^2-k^{-2})t] \tag{5}$$

and $(n, m) \in \mathbb{Z}^2$, $t \in \mathbb{R}$, k belongs to a subset $U \subset \mathbb{C}$ symmetrical with respect to $k=0$ and admitting $k=\infty$ and $k=0$ as boundary points, and $\hat{\varphi}(k, n, m, t)$ is a C^∞ function of t. The related set of functions $\hat{\varphi}(k, n, m, t)$ will be denoted by $\hat{\mathscr{F}}$. We also suppose that the following properties are satisfied.

(a) $\hat{\mathscr{F}}$ is a set of 'asymptotic' rational functions around ∞ and 0, with an 'asymptotic' unit element.

(b) $\mathscr{F}$ is a $\mathbb{C}$-linear space and is invariant under the action of the operators ξ, ξ^{-1}, η and ∂_t. (As a consequence $\mathscr{F}$ is a left $\mathscr{D}$-module where $\mathscr{D}$ is the ring of linear operators generated by $\{\gamma_0 + \gamma_1\xi + \gamma_2\xi^{-1} + \gamma_3\eta + \gamma_4\partial_t$, where $\gamma_j \in \mathbb{C}\}$).

(b′) $\mathscr{F}$ is invariant under the involution $k \to -k$.

More precisely (a) has the following meaning.

(i) Any $\hat{\varphi} \in \hat{\mathscr{F}}$ admits asymptotic expansions (AE) around ∞ and 0:

$$\hat{\varphi} \underset{\infty}{\sim} \sum_{q=-\infty}^{N} c_q k^q \qquad \hat{\varphi} \underset{0}{\sim} \sum_{q=-\infty}^{M} d_{-q} k^{-q} \tag{6}$$

where $N, M \in \mathbb{N}$ and the coefficients c_q, d_{-q} are functions of (n, m, t).

(ii) Any $\hat{\varphi} \in \hat{\mathscr{F}}$ is determined by the principal parts $\Sigma_{q=0}^{N} c_q k^q$ and $\Sigma_{q=1}^{M} d_{-q} k^{-q}$ of its AE at ∞ and 0. Note that (ii) generalises a property which is obvious for the ring $\mathscr{R}$ of (true) rational functions with possible poles at ∞ and 0 only. It is useful to associate with the asymptotic rational function $\hat{\varphi}$ its 'normalisation' defined as the (true) rational function in $\mathscr{R}$:

$$\widehat{\mathrm{Nor}}\, \hat{\varphi} \doteq \sum_{q=0}^{N} c_q k^q + \sum_{q=1}^{M} d_{-q} k^{-q}. \tag{7}$$

Then we can formulate (ii) in the form: the prjection $\widehat{\mathrm{Nor}}: \hat{\mathscr{F}} \to \mathscr{R}$ is one-to-one.

(iii) $\hat{\mathscr{F}}$ contains a privileged function $\hat{f}$ such that $\widehat{\mathrm{Nor}}\,\hat{f}=1$.

Observe that $f_0(-k)=(-1)^{n+m}f_0(k)$ (we omit the (n, m, t) dependence). Property (b′) implies that $g(k) \doteq f(-k) \in \mathscr{F}$ and $\widehat{\mathrm{Nor}}\,\hat{g}(k)=(-1)^{n+m}$. Since $\widehat{\mathrm{Nor}}(-1)^{n+m}\hat{f}(k)=(-1)^{n+m}$, it follows from (ii) that $g(k)=(-1)^{n+m}f(k)$, or equivalently $\hat{f}(-k)=\hat{f}(k)$. Therefore $\hat{f}$ has AE of the form

$$\hat{f} \underset{\infty}{\sim} 1+\sum_{q=1}^{\infty} a_{2q}k^{-2q} \qquad \hat{f} \underset{0}{\sim} b\left(1+\sum_{q=1}^{\infty} b_{2q}k^{2q}\right). \tag{8}$$

On the other hand, by comparing normalisations and using (ii) it is easy to see that every function of $\mathscr{F}$ can be written as a linear combination of the elements $\{\xi^q f\colon q \in \mathbb{Z}\}$ with coefficients being k-independent functions of (n, m, t).

In particular the expansion of ηf yields the equation

$$\left[\eta-\xi-\left(\frac{\eta b}{\xi^{-1}b}\right)\xi^{-1}\right]f=0. \tag{9}$$

By inserting the AE (8) of $\hat{f}$ into (9) we get

$$(\eta-\xi)a_2=\frac{\eta b}{\xi^{-1}b}-1 \tag{10a}$$

$$(\eta-\xi^{-1})b_2=\frac{\xi b}{\eta b}-1. \tag{10b}$$

In the same way $\partial_t f$ can be expanded in terms of $\{\xi^p f\colon -2 \leqslant p \leqslant 2\}$ and we obtain

$$\partial_t f=(1-\xi^2)a_2 f+\xi^2 f-\frac{b}{\xi^{-2}b}\,\xi^{-2}f. \tag{11}$$

By inserting the AE at $k=0$ of f into (11) and using (10a) we find

$$(\eta-\xi)\frac{\partial_t b}{b}=(1-\xi^2)\frac{\eta b}{\xi^{-1}b}+(\eta-\xi)(1-\xi^{-2})b_2. \tag{12}$$

Now equation (1) follows at once from (10b) and (12).

In conclusion any AM provides a solution to equation (1).

A natural way of constructing AM is to consider $\bar{\partial}$ equations [4, 5] of the form

$$\frac{\partial\psi(k)}{\partial\bar{k}}=\iint_{\mathbb{R}^2} r(k,l)\psi(l)\,\mathrm{d}l\wedge\mathrm{d}\bar{l} \qquad k\in\mathbb{C}-\{0,\mathrm{i},-\mathrm{i}\} \tag{13}$$

where $r(k, l)$ is a given function such that $\sup_{k\in\mathbb{C}}|r(k,l)|$ goes to zero fast enough as k tends to any of the values $0, \mathrm{i}, -\mathrm{i}, \infty$. Let $\mathscr{F}$ be the set of solutions $\varphi(k, n, m, t)$ to (13) which are bounded near i and −i and are such that the functions $\hat{\varphi} \doteq \varphi f_0^{-1}$ admit AE of the form (6) at 0 and ∞. By applying the generalised Cauchy formula one can prove that $\hat{\varphi}$ satisfies an integral equation $(1-J)\hat{\varphi}=\widehat{\mathrm{Nor}}\,\hat{\varphi}$ where the kernel of the integral operator J is determined by the distribution $r(k, l)$. With reasonable assumptions on $r(k, l)$ the operator $1-J$ turns out to be invertible and, as a consequence, $\mathscr{F}$ satisfies condition (ii) for AM. From its very definition $\mathscr{F}$ verifies conditions (i) and (b) too. Regarding (b′) it is enough to demand that $r(k, l)$ satisfies $r(k,l)=-r(-k,-l)$. Therefore, provided the solution f with $\widehat{\mathrm{Nor}}\,\hat{f}=1$ exists, the set $\mathscr{F}$ is an AM.

If $r(k, l)$ is a linear combination of delta functions the corresponding solutions of (1) can be computed easily. For example if we take

$$r(k, l) = \frac{\pi}{2\mathrm{i}} C(k, l)[\delta(k+k_0)\delta(l+l_0) - \delta(k-k_0)\delta(l-l_0)]$$

where k_0, $l_0 \in \mathbb{C}' \doteq \mathbb{C} - \{0, \mathrm{i}, -\mathrm{i}\}$ and

$$C(k, l) = -\frac{(kl)^{-1/2}}{2}\left(\frac{k+k^{-1}}{l+l^{-1}}\right)^{1/2}\left(k^2 - \frac{1}{k^2}\right)$$

then the integral equation for f reduces to a trivial linear system which leads to the one-soliton solution (3) with

$$\alpha = \log\left(\frac{k_0}{l_0}\right) \qquad \beta = \log\left(\frac{k_0 + k_0^{-1}}{l_0 + l_0^{-1}}\right)$$

$$\omega = \left(\frac{1}{k_0^2} + l_0^2\right) - \left(k_0^2 + \frac{1}{l_0^2}\right).$$

There are other possibilities for constructing AM (see the first preprint of [3]) which lead to interesting classes of solutions, such as the rational or finite-gap ones. An example of the strategy for obtaining rational solutions is the following. Given $\gamma \in \mathbb{C}'$ let $\mathcal{F}$ be the set of functions $\varphi(k, n, m, t) = \hat{\varphi}(k, n, m, t) f_0(k, n, m, t)$ such that $\hat{\varphi}(k)$ is analytic in $\mathbb{C} - \{\pm\gamma\}$ with at most single poles at $k = \pm\gamma$ and verifying $c_+ + c_- = 0$, $d_+ + d_- = 0$, where $c_\pm$, $d_\pm$ are the first coefficients of the Laurent expansions of φ around $k = \pm\gamma$:

$$\varphi(k) = \frac{c_\pm}{k \mp \gamma} + d_\pm + \mathcal{O}(k \mp \gamma) \qquad k \to \pm\gamma.$$

In addition we demand that $\hat{\varphi}$ admits asymptotic expansions of the form (6). It is easy to prove that $\mathcal{F}$ is an AM. The corresponding function f with unit normalisation is given by

$$f = \left(1 + \frac{W}{k^2 - \gamma^2}\right) f_0$$

with

$$W(n, m, t) = \frac{2\gamma^2[1 + (-1)^{n+m}]}{1 + 2[n + (\gamma - \gamma^{-1})(\gamma + \gamma^{-1})^{-1} m - 2(\gamma^2 + \gamma^{-2})t]}$$

which leads at once to the solution (4) of our model.

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References

[1] Volterra V 1931 *Leçons sur la Théorie Mathématique de la Lutte pour la Vie* (Paris: Gauthier-Villars)
[2] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[3] Jaulent M, Manna M and Martinez Alonso L 1988 *Asymptotic modules for solving non-linear equations* and *Scalar-bipolar asymptotic modules* and *Matrix-bipolar asymptotic modules. Preprints* Montpellier
[4] Jaulent M, Manna M and Martinez Alonso L 1988 *Inverse Problems* 4 123
[5] Jaulent M and Manna M 1986 *Europhys. Lett.* **2** 891